Problem Set 6

This sixth problem set explores the regular languages and their properties. This will be your first foray into computability theory, and I hope you find it fun and exciting!

As always, please feel free to drop by office hours, ask on Piazza, or email the staff list if you have any questions. We'd be happy to help out.

Good luck, and have fun!

Due Friday, May 19 at the start of class.

Problem One: Constructing DFAs

For each of the following languages over the indicated alphabets, construct a DFA that accepts precisely the strings that are in the indicated language. Your DFA does not have to have the fewest number of states possible.

Please use our online tool to design, test, and submit your answers to this problem. Handwritten or typed solutions will not be accepted. To use the tool, visit the CS103 website and click the "DFA/NFA Editor" link under the "Resources" header. If you're planning on submitting this assignment in a pair, in your GradeScope submission, please let us know the SUNetID (e.g. htiek, annasaps) of the partner who submitted the DFAs so that we can match the problem set to the submitted answers.

As a note, we will use an autograder to check your answers for this problem, so ***be sure to test your solutions before you submit!***

i. For the alphabet $\Sigma = \{$a, b, c$\}$, construct a DFA for the language $\{ w \in \Sigma^* \mid w$ contains exactly two cs. $\}$

ii. For the alphabet $\Sigma = \{$a, b$\}$, construct a DFA for the language $\{ w \in \Sigma^* \mid w$ contains the same number of instances of the substring ab and the substring ba $\}$. Note that substrings are allowed to overlap, so aba $\in L$ and babab $\in L$.

iii. For the alphabet $\Sigma = \{$a, b, c, …, z$\}$, construct a DFA for the language $\{ w \in \Sigma^* \mid w$ contains the word "cocoa" as a substring $\}$. *

iv. Suppose that you are taking a walk with your dog along a straight-line path. Your dog is on a leash that has length two, meaning that the distance between you and your dog can be at most two units. You and your dog start at the same position. Consider the alphabet $\Sigma = \{$y, d$\}$. A string in Σ^* can be thought of as a series of events in which either you or your dog moves forward one unit. For example, the string "yydd" means that you take two steps forward, then your dog takes two steps forward. Let $L = \{ w \in \Sigma^* \mid w$ describes a series of steps that ensures that you and your dog are never more than two units apart $\}$. Construct a DFA for L.

Problem Two: Constructing NFAs

For each of the following languages over the indicated alphabets, construct an NFA that accepts precisely the strings that are in the indicated language. ***Please use our online system to design, test, and submit your automata***; see above for details. As before, ***please test your submissions thoroughly!***

i. For the alphabet $\Sigma = \{$a, b, c$\}$, construct an NFA for the language $\{ w \in \Sigma^* \mid w$ ends in a, bb, or ccc $\}$.

ii. For the alphabet $\Sigma = \{$a, b, c, d, e$\}$, construct an NFA for the language $\{ w \in \Sigma^* \mid$ the last character of w appears nowhere else in w, and $|w| \geq 1 \}$.

iii. For the alphabet $\Sigma = \{$a, b$\}$, construct an NFA for the language $\{ w \in \Sigma^* \mid w$ contains at least two b's with exactly five characters between them $\}$. For example, baaaaab is in the language, as is aabaabaaabbb and abbbbbabaaaaaab, but bbbbb is not, nor are bbbab or aaabab.

* DFAs are often used to search large blocks of text for specific substrings, and several string searching algorithms are built on top of specially-constructed DFAs. The *Knuth-Morris-Pratt* and *Aho-Corasick* algorithms use slightly modified DFAs to find substrings extremely efficiently.

Problem Three: $\wp(\Sigma^*)$

Let Σ be an alphabet. Give a short English description of the set $\wp(\Sigma^*)$. Briefly justify your answer. *(We think that there is a single "best answer." You should be able to describe the set in at most ten words)*

Problem Four: Concatenation, Kleene Stars, and Complements

The regular languages are closed under a number of different operations. This problem explores some properties of those operations.

i. Prove or disprove: if L is a nonempty, finite language and k is a positive natural number, then $|L^k| = |L|^k$. Here, the notation $|L|^k$ represents "the cardinality of L, raised to the kth power," and the notation $|L^k|$ represents "the cardinality of the k-fold concatenation of L with itself."
ii. Prove or disprove: there is a language L where $\overline{(L^*)} = (\overline{L})^*$.

Problem Five: Arden's Lemma

When you were first learning algebra, you probably learned a family of techniques to solve equations in which a variable x was on both sides of an equals sign. For example, you probably learned how to look at a formula like

$$x^2 = ax + b$$

and to use the quadratic formula to solve for x.

It's also possible to set up equations involving some unknown that appears on both sides of an equals sign, but where the quantities involved are *languages* rather than numbers. For example, if A and B are languages, you may want to determine what languages X satisfy the equality

$$X = AX \cup B.$$

Just as the quadratic formula is a useful tool for solving for x given a quadratic equation, in formal language theory there's a result called ***Arden's lemma*** that's useful for solving for X in an equation of the above form. Specifically, Arden's lemma says that, given the equality $X = AX \cup B$, you are guaranteed that

$$A^*B \subseteq X.$$

In this problem, we're going to ask you to prove Arden's lemma.

Let's begin with a refresher of the key terms and definitions involved. As a reminder, if L_1 and L_2 are languages over an alphabet Σ, then the ***concatenation of L_1 and L_2***, denoted L_1L_2, is the language

$$L_1L_2 = \{\ wx \mid w \in L_1 \text{ and } x \in L_2\ \}.$$

From concatenation, we can define ***language exponentiation*** of a language L inductively as follows:

$$L^0 = \{\varepsilon\} \qquad\qquad L^{n+1} = LL^n$$

You may find these formal terms helpful in the course of solving this problem.

i. Let A and B be arbitrary languages over some alphabet Σ. Prove, by induction, that if $X = AX \cup B$, then $A^nB \subseteq X$ for every $n \in \mathbb{N}$. Please use the formal definitions of concatenation, language exponentiation, union, and subset in the course of writing up your answer.

If you'll recall, we formally defined the ***Kleene closure*** of a language L over Σ to be the language

$$L^* = \{\ w \in \Sigma^* \mid \text{there is some } n \in \mathbb{N} \text{ such that } w \in L^n\ \}.$$

ii. Let A and B be arbitrary languages over some alphabet Σ. Using your result from part (i) of this problem and the formal definition of L^*, prove that if $X = AX \cup B$, then $A^*B \subseteq X$.

Problem Six: Hard Reset Sequences

A ***hard reset sequence*** for a DFA is a string *w* with the following property: starting from any state in the DFA, if you read *w*, you end up in the DFA's start state.

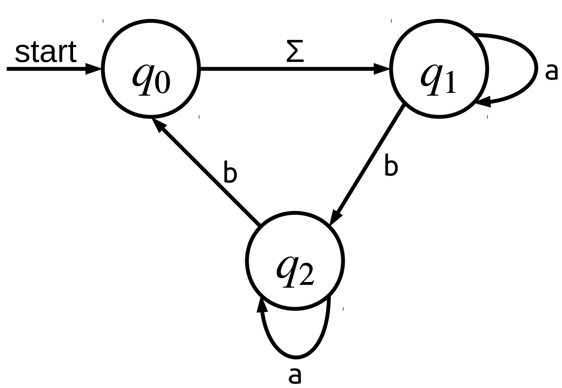
Hard reset sequences have many practical applications. For example, suppose you're remotely controlling a Mars rover whose state you're modeling as a DFA. Imagine there's a hardware glitch that puts the Mars rover into a valid but unknown state. Since you can't physically go to Mars to pick up the rover and fix it, the only way to change the rover's state would be to issue it new commands. To recover from this mishap, you could send the rover a hard reset sequence. Regardless of what state the rover got into, this procedure would guarantee that it would end up in its initial configuration.

Here is an algorithm that, given any DFA, will let you find every hard reset sequence for that DFA:

1. Add a new start state q_s to the automaton with ε-transitions to every state in the DFA.
2. Perform the subset construction on the resulting NFA to produce a new DFA called the ***power automaton***.
3. If the power automaton contains a state corresponding solely to the original DFA's start state, make that state the only accepting state in the power automaton. Otherwise, make every state in the power automaton a rejecting state.

This process produces a new automaton that accepts all the hard reset sequences of the original DFA. It's possible that a DFA won't have any hard reset sequences (for example, if it contains a dead state), in which case the new DFA won't accept anything.

Apply the above algorithm to the following DFA and give us a hard reset sequence for that DFA. For simplicity, please give the subset-constructed DFA as a transition table rather than a state-transition diagram. We've given you space for the table over to the right, and to be nice, we've given you exactly the number of rows you'll need.



	a	b

Sample hard reset sequence: ________________________

Problem Seven: Complementing NFAs

In lecture, we saw that if you take a DFA for a language L and flip all the accepting and rejecting states, you end up with a DFA for $\overline{L}$.

Draw a simple NFA for a language L where flipping all the accepting and rejecting states does not produce an NFA for $\overline{L}$. Briefly justify your answer; you should need at most a sentence or two here.

Problem Eight: DFAs, Formally

When we first talked about graphs, we saw them first as pictures (objects connected by lines), but then formally defined a graph G as an ordered pair (V, E), where V is a set of nodes and E is a set of edges. This rigorous definition tells us what a graph actually is in a mathematical sense, rather than just what it looks like.

We've been talking about DFAs for a while now and seen how to draw them both as a collection of states with transitions (that is, as a state-transition diagram) and as a table with rows for states and columns for characters. But what exactly *is* a DFA, in a mathematical sense?

Formally speaking, a DFA is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

- Q is a finite set, the elements of which we call *states*;
- Σ is a finite, nonempty set, the elements of which we call *characters*;
- $\delta : Q \times \Sigma \to Q$ is the ***transition function***, described below;
- $q_0 \in Q$ is the start state;
- $F \subseteq Q$ is the set of accepting states.

The transition function warrants a bit of explanation. When we've drawn DFAs, we've represented the transitions either by arrows labeled with characters or as a table with rows and columns corresponding to states and symbols, respectively. In this formal definition, the transition function δ is what ultimately specifies the transition. Specifically, for any state $q \in Q$ and any symbol $a \in \Sigma$, the transition from state q on symbol a is given by $\delta(q, a)$.

This question explores some properties of this rigorous definition.

i. Is it possible for a DFA to have no states? If so, define a DFA with no states as a 5-tuple, explaining why your 5-tuple meets the above requirements. If not, explain why not.

ii. Is it possible for a DFA to have no *accepting* states? If so, define a DFA with no accepting states as a 5-tuple, explaining why your 5-tuple meets the above requirements. If not, explain why not.

iii. In class, we said that a DFA must obey the rule that for any state and any symbol, there has to be exactly one transition defined on that symbol. What part of the above definition guarantees this?

iv. Is it possible for a DFA to have a state that can't ever be reached (that is, a state that can't ever be entered by any string)? If so, define a DFA with an unreachable state as a 5-tuple, explaining why your 5-tuple meets the above requirements. If not, explain why not.

Going forward, in CS103, we won't use the formal definition of DFAs in our proofs, not because it's not useful, but because it often makes the reasoning a bit harder to follow. However, we thought you should at least see the definition, since it's useful for formalizing what a DFA actually is!

Problem Nine: Why the Extra State?

In our proof that the regular languages are closed under the Kleene closure operator (that is, if L is regular, then L^* is regular), we used the following construction:

1. Begin with an NFA N where $\mathscr{L}(N) = L$.
2. Add in a new start state q_{start}.
3. Add an ε-transition from q_{start} to the start state of N.
4. Add ε-transitions from each accepting state of N to q_{start}.
5. Make q_{start} an accepting state.
6. Make every state besides q_{start} a rejecting state.

You might have wondered why we needed to add q_{start} as a new state to the NFA. It might have seemed more natural to do the following:

1. Begin with an NFA N where $\mathscr{L}(N) = L$.
2. Add ε-transitions from each accepting state of N to the start state of N.
3. Make the start state of N an accepting state.
4. Make every other state of N a rejecting state.

Unfortunately, this construction does not work correctly.

Find a regular language L and an NFA N for L such that using the second construction does not create an NFA for L^*. Justify why the language of the new NFA isn't L^*.

Extra Credit Problem: Why Finite? (1 Point Extra Credit)

The term "finite" in finite automata refers to the fact that each automaton can only have finitely many states. It turns out there's a good reason for that.

We'll say that a ***deterministic infinite automaton***, or ***DIA***, is a generalization of a DFA in which the automaton has infinitely many different states. Formally speaking, a DIA is given by the same 5-tuple definition as a DFA from Problem Eight, except that Q must be an infinite set. Since DIAs have infinitely many states, they can't actually be built, and so are mostly an object of purely theoretical study.

Prove that if L is an arbitrary language over an alphabet Σ, then there is a DIA that accepts L (that is, the DIA accepts every string in L and rejects every string not in L.)